Design of Nonlinear PID Controllers

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A typical approach to nonlinear process control problems is to base the controller design on a linearized model of the plant about a nominal set point (constant steady state). Such linear controller designs can be expected to perform satisfactorily as long as the plant is operated in a range sufficiently close to that set point. When the plant is to be operated over a wider range, this procedure can be repeated at a number of set points, and the controller can be retuned as operating conditions change. Of course, there has been considerable recent interest in automating this procedure by means of the concepts of self-tuning or adaptive controllers. These approaches appear to be particularly valuable in the situation where the plant model is poorly known.

In recent years another approach has been developed that is well suited for the case in which an accurate model for the nonlinear plant, or at least its family of linearizations, is available. The nonlinear plant is assumed to have a continuous, parameterized family of set points, and the dynamics are represented by the corresponding parameterized family of linearizations. Based on this information, a parameterized linear controller is designed so that at each value of the parameter, that is, at each set point, the closed-loop linearization has the desired characteristics. Then a nonlinear controller is computed which is such that if it is linearized about any set point, the designed linear controller for that set point is obtained. Therefore the nonlinear closed-loop system should perform well when operated near any set point in the family. Nonlocal performance typically is evaluated by simulation.

To date this approach, called *design by extended linearization,* has been studied mainly within the framework of state variable models and the eigenvalue placement problem, as in the work of Baumann and Rugh (1986). (A related approach by Reboulet and Champetier, 1984, called design by pseudolinearization also should be noted.) To demonstrate that the extended linearization methodology can address a large class of process control problems, attention will be focused on the design of nonlinear proportional-integral-derivative (PID) controllers based on a parameterized version of the well-known Ziegler-Nichols tuning specifications.

In recent years, control schemes have been proposed that involve nonlinear gain weightings of various types. Examples can be found in Shinskey (1979) and Cheung and Luyben (1980). The extended linearization approach differs significantly from these methods in that the nonlinear characteristics of the plant, as reflected in the set point and linearization families, are explicitly used to specify the nonlinear gains.

Design Method

Assume that a nonlinear plant model and nonlinear sensor model are given in the standard interconnection shown in Figure 1. The objective is to find a nonlinear controller such that at every closed-loop set point the linearization of the nonlinear controller is the appropriate PID controller for the linearization of the plant and sensor models. Of course, the linearizations of the subsystems in Figure 1 depend upon the set point. Denoting set point signal values with boldface, the linearized closed-loop system is shown in Figure 2, where the plant and controller linearizations are parameterized by constant values $\mathbf{u}$ of the plant input signal $u(t)$.

It is assumed that for the range of set points (values of $\mathbf{u}$) of interest, the plant linearization transfer function has the form

$$G_{\mathbf{u}}(s) = e^{-sT_1(\mathbf{u})} \frac{c_{n-1}(\mathbf{u})s^{n-1} + \cdots + c_0(\mathbf{u})}{s^n + a_{n-1}(\mathbf{u})s^{n-1} + \cdots + a_0(\mathbf{u})} \tag{1}$$

where all coefficients are continuous functions of $\mathbf{u}$, $T_1(\mathbf{u}) \geq 0$, and $a_0(\mathbf{u}) \neq 0$. Thus, for all $\mathbf{u}$, the parameterized plant linearization has no pole at $s = 0$, and the set point output is given by

$$\mathbf{y}(\mathbf{u}) = \int_0^{\mathbf{u}} G_\sigma(0)\, d\sigma = \int_0^{\mathbf{u}} \frac{c_0(\sigma)}{a_0(\sigma)}\, d\sigma \tag{2}$$

The sensor linearization transfer function is assumed to have the form

$$H_{\mathbf{y}(\mathbf{u})}(s) = e^{-sT_2[\mathbf{y}(\mathbf{u})]} \frac{b_{n-1}[\mathbf{y}(\mathbf{u})]s^{n-1} + \cdots + b_0[\mathbf{y}(\mathbf{u})]}{s_n + d_{n-1}[\mathbf{y}(\mathbf{u})]s^{n-1} + \cdots + d_0[\mathbf{y}(\mathbf{u})]} \tag{3}$$

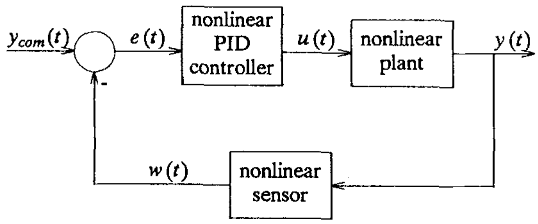

Figure 1. Nonlinear closed-loop system.

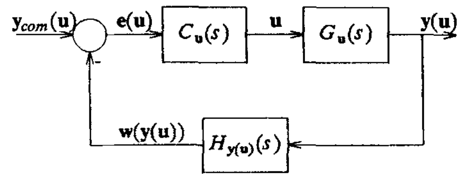

Figure 2. Linearized closed-loop system with set point values.

where, again, all coefficients are continuous, $T_2[y(u)] \geq 0$, and $b_0[y(u)] = d_0[y(u)] \neq 0$. Thus the parameterized sensor linearization has no pole at $s = 0$, and the set point sensor output is identical to the set point sensor input, $w[y(u)] = y(u)$.

The controller linearization family will be chosen to have the PID form

$$C_u(s) = K_1(u) + \frac{K_2(u)}{s} + K_3(u)s \qquad (4)$$

where the parameterized gains remain to be determined, subject to the constraints that they are continuous in u and that $K_2(u)$ is always nonzero. Due to the integral term in the controller, $e(u) = 0$ at any set point u. Thus $y(u) = y_{com}(u)$, and the closed-loop system will have zero offset at each set point.

The first step in the design method is to specify the parameterized controller gains over the range of set points of interest. Assuming the typical situation where at each set point a proportional-gain controller yields a stable closed-loop system for sufficiently small gain, the Ziegler-Nichols approach can be used. This involves determining expressions for the so-called ultimate gain $K_o(u)$ and ultimate period $P_o(u)$. Recall that in terms of the open-loop transfer function $G_u(s)H_{y(u)}(s)$, the phase crossover frequency $\omega_o(u)$ is related to the ultimate period via $P_o(u) = 2\pi/\omega_o(u)$, and the ultimate gain is the reciprocal of the magnitude of the open-loop transfer function at this frequency.

In most cases it will not be possible to obtain exact expressions for these quantities over the range of set points of interest. This might be due to the complexity of the transfer functions or, in the absence of a nonlinear plant model, because values of $\omega_o(u)$ and $K_o(u)$ have been obtained by experiments on the plant in the neighborhood of various set points. In these situations, approximate expressions must be obtained by curve fitting or other means. These aspects will be illustrated in an example in the sequel, so for now it will be assumed that suitable expressions for $K_o(u)$ and $\omega_o(u)$ have been determined. Then the Ziegler-Nichols settings for the linearized controller gains are

$$K_1(u) = 0.6K_o(u), \quad K_2(u) = \frac{0.6}{\pi}K_o(u)\omega_o(u), \quad K_3(u) = \frac{0.6\pi K_o(u)}{4\omega_o(u)} \qquad (5)$$

The second step in the design process is to choose a nonlinear controller whose linearization about each set point u is described by the transfer function $C_u(s)$ in Eq. 4 specified by Eq. 5. Of course there are many nonlinear controllers that will satisfy this objective, as discussed by Wang and Rugh (1987). A simple choice is the nonlinear controller described by the scalar state equation

$$\dot{x}(t) = K_2[x(t)]e(t)$$
$$u(t) = x(t) + K_1[x(t)]e(t) + K_3[x(t)]\dot{e}(t) \qquad (6)$$

This nonlinear controller has the set point family specified by $e(u) = 0$, $x(u) = u$, and a simple computation shows that the controller linearization family has the desired, parameterized transfer function $C_u(s)$. In some cases, including the following example, it is possible to solve the differential equation in Eq. 6 analytically, and express the controller as an explicit, nonlinear function of $e(t)$, its derivative, and integral. Whether this results in a simpler implementation is problematic.

First Example

Consider a series of n identical, noninteracting tanks, where the input $u(t)$ is the (nonnegative) flow into the first tank, and the output $y(t)$ is the liquid height in the nth tank. Denoting the height in the ith tank by $h_i(t)$, a standard model for this plant takes the form

$$\dot{h}_1(t) = \frac{-c}{A}h_1^{1/2}(t) + \frac{1}{A}u(t)$$
$$\dot{h}_i(t) = \frac{-c}{A}h_i^{1/2} + \frac{c}{A}h_{i-1}^{1/2}(t), \quad i = 2, \ldots, n$$
$$y(t) = h_n(t) \qquad (7)$$

Here $A > 0$ is the cross-sectional area of the tanks, and $c > 0$ is the outflow resistance. For constant input, $u(t) = u > 0$, the set point values are given by

$$h_i(u) = \frac{u^2}{c^2}, \quad i = 1, \ldots, n, \quad y(u) = \frac{u^2}{c^2} \qquad (8)$$

About any such set point, the linearized plant is

$$\dot{h}_{\delta 1}(t) = \frac{-c^2}{2Au}h_{\delta 1}(t) + \frac{1}{A}u_\delta(t)$$
$$\dot{h}_{\delta i}(t) = \frac{-c^2}{2Au}h_{\delta i}(t) + \frac{c^2}{2Au}h_{\delta,i-1}(t), \quad i = 2, \ldots, n$$
$$y_\delta(t) = h_{\delta n}(t) \qquad (9)$$

where the deviation variables are defined by

$$u_\delta(t) = u(t) - u, \quad h_{\delta i}(t) = h_i(t) - \frac{u^2}{c^2}, \quad i = 1, \ldots, n, \quad y_\delta(t) = y(t) - \frac{u^2}{c^2} \qquad (10)$$

A simple computation shows that the set point family and corresponding parameterized transfer function for the plant can be written in the form

$$G_u(s) = \frac{K/u^{n-1}}{(s + \tau/u)^n}, \quad y(u) = \frac{K}{2\tau^n}u^2; \quad K, \tau > 0 \tag{11}$$

where

$$K = \frac{1}{A}\left(\frac{c^2}{2A}\right)^{n-1}, \quad \tau = \frac{c^2}{2A} \tag{12}$$

For simplicity, it is assumed that the sensor is linear with unity transfer function.

To compute the Ziegler-Nichols gain for the PID controller design, it is assumed that $n \geq 3$. The phase crossover frequency and the ultimate gain for the linearized plant are specified by

$$\text{Arg } G_u[j\omega_o(u)] = -\pi, \quad K_o(u) = \frac{1}{|G_u[j\omega_o(u)]|} \tag{13}$$

Simple calculations show that the first condition implies

$$\omega_o(u) = \frac{\tau \tan(\pi/n)}{u} \tag{14}$$

and then

$$K_o(u) = \frac{\tau^n \sec^n(\pi/n)}{Ku} \tag{15}$$

Computing the parameterized Ziegler-Nichols gain leads to the nonlinear controller

$$\dot{x}(t) = \frac{0.6\tau^{n+1}\tan(\pi/n)\sec^n(\pi/n)e(t)}{K\pi x^2(t)}$$

$$u(t) = x(t) + \frac{0.6\tau^n \sec^n(\pi/n)e(t)}{Kx(t)} + \frac{0.15\pi\tau^{n-1}\sec^n(\pi/n)\dot{e}(t)}{K\tan(\pi/n)} \tag{16}$$

To illustrate the performance of this controller, suppose $n = 3$, $\tau = 1$, and $K = 2$. The nonlinear controller is then

$$\dot{x}(t) = \frac{1.32e(t)}{x^2(t)}$$

$$u(t) = x(t) + \frac{2.40e(t)}{x(t)} + 1.09\dot{e}(t) \tag{17}$$

This will be compared to the Ziegler-Nichols linear PID controller designed for the nominal set point specified by $u = 7$, $y(u) = 49$, that is, the linear controller described by

$$C(s) = 0.34 + \frac{0.03}{s} + 1.09s \tag{18}$$

Simulation of the nonlinear plant with both the linear and nonlinear controllers yields the following results. When the system is at rest near the nominal set point $u = 7$, the responses of the linearly-controlled and nonlinearly-controlled systems are very similar for small step-function changes in the set point command input. For larger step commands, the characteristics of the linearly-controlled system begin to change as shown in Figure 3. For step commands to the system at rest at the set point $u = 10$, the linear controller provides too little damping, as shown in Figure 4. On the other hand, from the set point corresponding to $u = 4$, the linear controller yields a more damped response, as shown in Figure 5. In all cases, the nonlinear controller maintains the basic Ziegler-Nichols response characteristics.

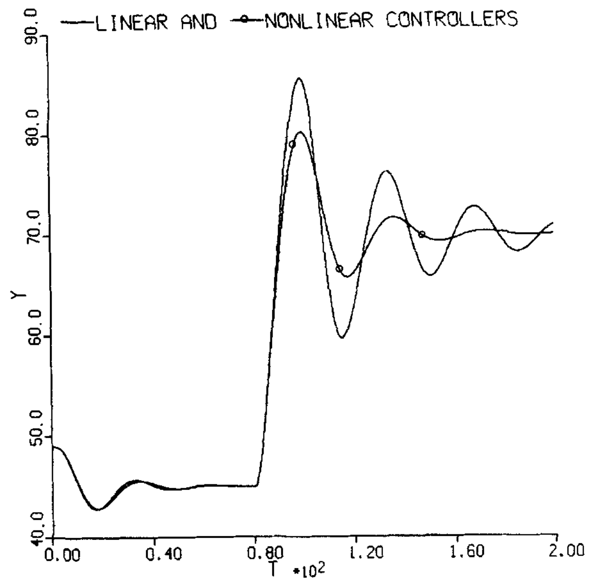

Figure 3. Responses for step-function set point changes from u = 7.

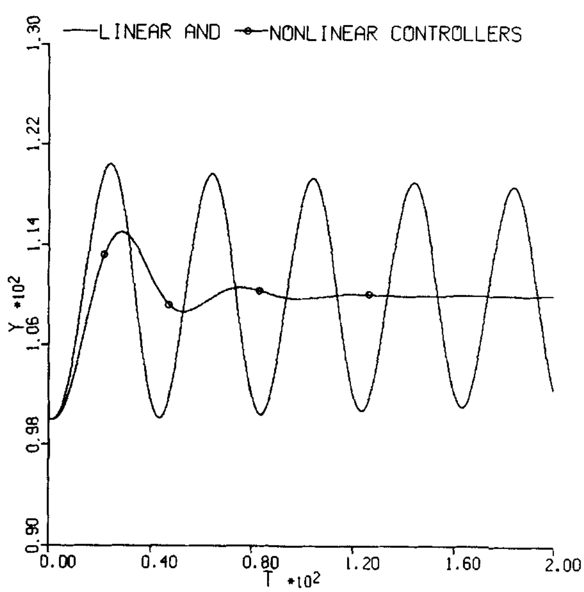

Figure 4. Responses for step-function set point changes from u = 10.

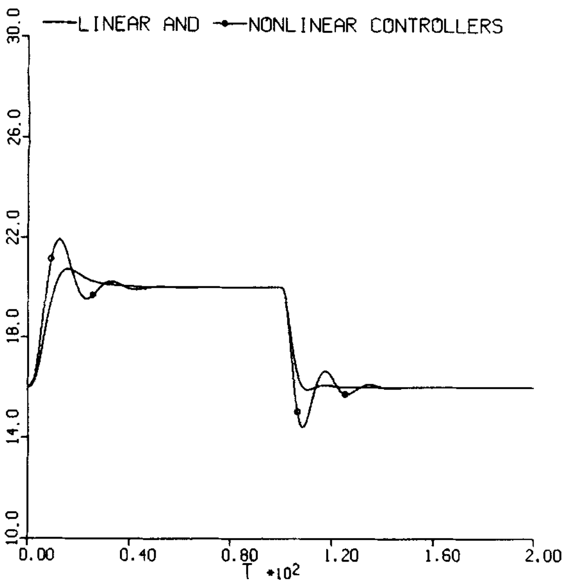

Figure 5. Responses for step-function changes from $u = 4$.

Second Example

Two identical, noninteracting tanks with a constant transport lag give rise to the following plant set point family and parameterized transfer function:

$$y(u) = \frac{K}{2\tau^2}u^2, \quad G_u(s) = \frac{K/u}{(s + \tau/u)^2}e^{-sT}, \qquad K, \tau, T > 0 \tag{19}$$

In this case $\omega_o(u)$ is specified by

$$-\omega_o(u)T - 2\tan^{-1}\left[\frac{\omega_o(u)}{\tau/u}\right] = -\pi \tag{20}$$

so that an analytical expression for $\omega_o(u)$ is unavailable and an approximation must be found. To simplify notation, let

$$x = \frac{T\tau}{2u}, \quad y = \frac{T\omega_o(u)}{2} \tag{21}$$

Then, after some manipulation, the transcendental expression, Eq. 20, can be written in the form

$$y \tan(y) = x \tag{22}$$

Assuming a set point range of interest where u is sufficiently large that x is small, then the following series approximations are reasonable. Using the Taylor series for tan (y) gives

$$y^2 + \frac{1}{3}y^4 + \frac{2}{15}y^6 + \cdots = x \tag{23}$$

and reversion of the series (in y^2) yields

$$y = \left(x - \frac{1}{3}x^2 + \frac{4}{45}x^3 - \cdots\right)^{1/2} \tag{24}$$

In terms of the original variables, Eq. 24 becomes

$$\omega_o(u) = \frac{2}{T}\left(\frac{T\tau}{2u} - \frac{T^2\tau^2}{12u^2} + \frac{T^3\tau^3}{90u^3} - \cdots\right)^{1/2} \tag{25}$$

Then the ultimate gain is given by

$$K_o(u) = \frac{\tau^2 + u^2\omega_o^2(u)}{Ku} \tag{26}$$

Simulation studies show, for example, that a linear controller designed for the set point $u = 7$ yields unstable, oscillatory responses to small, step-function set point commands when the system is at rest at the set point $u = 10$. A nonlinear controller implemented with $\omega_o(u)$ truncated to the first term in the expansion yields satisfactory response characteristics to step-function commands increasing the set point from $u = 10$, though only small step-function commands reducing the set point yield satisfactory responses.

Concluding Remarks

Although this paper has concentrated on the Ziegler-Nichols specifications for concreteness, it should be clear that other classical linear design rules can be extended to the nonlinear setting in a similar fashion. Also, modifications of the PID structure that avoid pure differentiation and pure integration can be extended.

The nonlinear controller designs used here work best for smooth, slowly-varying command inputs, for such signals will not drive the variables of the nonlinear closed-loop system far from the collection of possible set point values. This intuition is supported by the local stability result reported by Kelemen (1986). Indeed, the step-function commands used in the example simulations disfavor the design approach.

A major advantage of design by extended linearization is that it permits direct application of the large body of linear design knowledge in setting specifications for, and predicting the characteristics of, the nonlinear closed-loop system. Also, coordinate changes are not required as in a number of other modern nonlinear approaches, so the design is carried out in the original plant-description variables. Of course, the approach is inherently local in nature; just as with a linear controller, nonlocal response characteristics, including stability, of the nonlinear closed-loop system typically are investigated by means of simulation.

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Notation

A = cross-sectional area
c = outflow resistance constant
$C_u(s)$ = linearized compensator transfer function
$e(t)$; e = error signal; set point value
$G_u(s)$ = linearized plant transfer function
$h_i(t)$ = liquid height in ith tank
$H_y(s)$ = linearized sensor transfer function
K = gain constant
$K_o(u)$ = parameterized ultimate gain
$K_1(u)$ = parameterized proportional gain
$K_2(u)$ = parameterized integrator gain

$K_3(u)$ – parameterized differentiator gain
PID – proportional-integral-derivative
T_1, T_2, T – delay times
$u(t)$; u – plant input; set point value
$w(t)$; w – sensor output; set point value
$x(t)$ – nonlinear controller state
$y(t)$; y – plant output; set point value
$y_{com}(t)$; y_{com} – command input; set point value
τ – time constant
$\omega_o(u)$ – parameterized ultimate frequency

Literature Cited

Baumann, W. T., and W. J. Rugh, "Feedback Control of Nonlinear Systems by Extended Linearization," *IEEE Trans. Auto. Control,* **31**(1), 40, (1986).

Cheung, T. F., and W. L. Luyben, "Nonlinear and Nonconventional Liquid Level Controllers," *Ind. Eng. Chem. Fundam.,* **19**(1), 93, (1980).

Kelemen, M., "A Stability Property," *IEEE Trans. Auto. Control,* **31**(8), 766, (1986).

Reboulet, C., and C. Champetier, "A New Method for Linearizing Nonlinear Systems: The Pseudolinearization," *Int. J. Control,* **40**(4), 631, (1984).

Shinskey, F. G., *Process Control Systems,* McGraw-Hill, New York, (1979).

Wang, J. L., and W. J. Rugh, "Parameterized Linear Systems and Linearization Families for Nonlinear Systems," *IEEE Trans. Circuits, Systems,* **34**(6), 650 (1987).

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